The Japanese Economic Review
The Journal of the Japanese Economic Association

The Japanese Economic Review
Vol. 63, No. 1, March 2012

doi: 10.1111/j.1468-5876.2011.00548.x

QUANTILE FORECASTS OF FINANCIAL RETURNS USING REALIZED GARCH MODELS*

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This article applies the realized generalized autoregressive conditional heteroskedasticity (GARCH) model, which incorporates the GARCH model with realized volatility, to quantile forecasts of financial returns, such as Value-at-Risk and expected shortfall. Student's *t*- and skewed Student's *t*-distributions as well as normal distribution are used for the return distribution. The main results for the S&P 500 stock index are: (i) the realized GARCH model with the skewed Student's *t*-distribution performs better than that with the normal and Student's *t*-distributions and the exponential GARCH model using the daily returns only; and (ii) using the realized kernel to take account of microstructure noise does not improve the performance.
JEL Classification Numbers: C52, C53, G17.

1. Introduction

Quantile forecasts of financial returns are important for financial risk management, such as Value-at-Risk (VaR) and expected shortfall (ES). Few would dispute the fact that financial volatility changes over time and hence it is important to model the dynamics of volatility. One of the most widely used is the autoregressive conditional heteroskedasticity (ARCH) family, including the ARCH model by Engle (1982), the generalized ARCH (GARCH) model by Bollerslev (1986) and their extensions. Recently, realized volatility has also attracted the attentions of financial econometricians as an accurate estimator of volatility.

There are two problems in calculating realized volatility. First, realized volatility is influenced by market microstructure noise, such as bid-ask spread and non-synchronous trading (Campbell *et al*., 1997). There are some methods available for mitigating the effect of microstructure noise on realized volatility (Zhang *et al*., 2005; Bandi and Russell, 2008, 2011; Barndorff-Nielsen *et al*., 2008, 2011). Second, there are non-trading hours, such as overnight and lunch-time, when we cannot obtain high-frequency returns. Adding the squares of overnight returns may make realized volatility noisy. Hansen and Lunde (2005a,b) propose a method for calculating realized volatility without overnight returns.

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* I would like to thank the Associate Editor and the anonymous referee for their helpful comments and suggestions. I would also like to thank Kazuo Ogawa, Kosuke Oya, Yasuhiro Omori and the participants in the 2011 Japan Economic Association Annual Meeting at Kumamoto Gakuen University and the Stanford Institute for Theoretical Economics (SITE) Summer 2011 Workshop Segment 1 "Measuring and Modeling Risk with High Frequency Data" for their constructive comments. Financial support from the Ministry of Education, Culture, Sports, Science and Technology of the Japanese Government through Grant-in-Aid for Scientific Research (No.18203901; 21243018; 22243021), the Global COE program "Research Unit for Statistical and Empirical Analysis in Social Sciences" at Hitotsubashi University and the Joint Usage and Research Center, Institute of Economic Research, Hitotsubashi University (IERPK1109) is gratefully acknowledged. Thanks are also due to Makoto Takahashi and Asuka Takeuchi-Nogimori for checking the article. All remaining errors are solely my responsibility.

Hansen *et al*. (2011) have recently proposed to extend GARCH models incorporating them with realized volatility. Their models, which are called realized GARCH models, have certain advantages in quantile forecasts. First, they can adjust the bias of realized volatility caused by microstructure noise and non-trading hours. Second, they enable us to estimate the parameters of return and volatility equations simultaneously. Thus, we can estimate the parameters of the return distribution jointly with the other parameters of the model. Takahashi *et al*. (2009) have extended the stochastic volatility (SV) model in the same direction. The estimation of the realized GARCH model is less time-consuming than that of the realized SV model because the former can be estimated by the maximum likelihood method while the latter requires more computer-intensive methods, such as simulated maximum likelihood estimation via importance sampling and Bayesian estimation via Markov chain Monte Carlo (MCMC). In this article, we apply the realized GARCH model to quantile forecasts. GARCH models and realized volatility have already been applied to quantile forecasts (Giot and Laurent, 2004; Watanabe and Sasaki, 2006; Clements *et al*., 2008) but to the best of my knowledge this paper is the first to apply the realized GARCH model to quantile forecasts.

In this article, we use the Student's *t*- and skewed Student's *t*-distributions as well as the normal distribution for the return distribution because it is straightforward to estimate the parameters in the Student's *t*- and skewed Student's *t*-distributions jointly with the other parameters in the realized GARCH model by the maximum likelihood method. If the realized GARCH model can adjust the bias of realized volatility caused by microstructure noise correctly, we need not take the bias into account in calculating realized volatility. To analyze whether it is true, we use the plain realized volatility, which is the sum of the squared intraday returns and the realized kernel proposed by Barndorff-Nielsen *et al*. (2008) to take account of microstructure noise. For comparison, we also use the exponential GARCH (EGARCH) model proposed by Nelson (1991), which is estimated using daily returns only. The data we use are daily returns, realized volatility and realized kernel of the S&P 500 stock index. The main results are: (i) the realized GARCH model with the skewed Student's *t*-distribution performs better than that with normal and Student's *t*-distributions and the EGARCH model using the daily returns only; and (ii) the performance does not improve if the realized kernel, which takes account of microstructure noise, is used instead of the plain realized volatility, implying that the realized GARCH model can adjust the bias caused by microstructure noise.

The article proceeds as follows. Section 2 reviews the realized GARCH model. Section 3 explains the method for forecasting the 1-day-ahead VaR and ES using the realized GARCH model. Section 4 explains the data and summarizes the empirical results. Conclusions and possible extensions are given in Section 5.

2. Realized GARCH model

We start with a brief review of the realized GARCH model. Daily return R_t is specified as

$$R_t = \mathrm{E}(R_t|\boldsymbol{I}_{t-1}) + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad z_t \sim i.i.d.(0,1), \tag{1}$$

where $\mathrm{E}(R_t|\boldsymbol{I}_{t-1})$ is the expectation of R_t conditional on the information up to day $t-1$, σ_t^2 is the volatility, and z_t is the standardized error which follows an independent and identical distribution with mean 0 and variance 1. In what follows, we set $E(R_t|\boldsymbol{I}_{t-1}) = 0$ because the null hypothesis of zero mean and that of no autocorrelations are not rejected in our empirical application. The distribution of z_t will be explained below.

For volatility specification, we use the simplest version of realized GARCH models:

$$\ln\sigma_t^2 = \omega + \beta\ln\sigma_{t-1}^2 + \gamma X_{t-1}, \tag{2}$$

$$X_t = \mu + \varphi\ln\sigma_t^2 + \tau_1 z_t + \tau_2\left(z_t^2 - 1\right) + u_t, \quad u_t \sim i.i.d.\mathrm{N}\left(0, \sigma_u^2\right), \tag{3}$$

where X_t denotes the log of realized volatility.

Equation (2) specifies the dynamics of the true volatility σ_t^2. While GARCH models specify σ_t^2 as a function of the past values of σ_t^2 and ε_t (or z_t), the realized GARCH model specifies it as a function of the past values of σ_t^2 and X_t. Equation (3) is called the measurement equation, which relates the realized volatility to the true volatility. If the realized volatility were an unbiased estimator of the true volatility, μ and φ would be 0 and 1, respectively. Realized volatility, however, has a bias caused by microstructure noise and non-trading hours. For example, the New York Stock Exchange is open only for 6.5 h in a day. Suppose that R_t and σ_t^2 are return and volatility for a whole day and RV_t is realized volatility calculated using the intraday returns only when the market is open. Then, we should expect $\mu < 0$ or $\varphi < 1$. Equation (3) assumes that X_t, i.e., the log of realized volatility, depends on the current value of z_t. If $\tau_1 < 0$, X_t will be larger when $z_t < 0$ than when $z_t > 0$, which will make σ_{t+1}^2 larger when $z_t < 0$ through (2) if $\gamma > 0$. This is consistent with the well-known phenomenon in stock markets of a negative correlation between today's return and tomorrow's volatility.

The above model is the realzed GARCH (1, 1) model. The realized GARCH (p, q) model replaces (2) with

$$\ln\sigma_t^2 = \omega + \sum_{i=1}^{p}\beta_i \ln\sigma_{t-i}^2 + \sum_{j=1}^{q}\gamma_j X_{t-j}. \tag{4}$$

We also estimate the realized GARCH (1, 2) (2, 1) and (2, 2) models but the performance of quantile forecasts does not change so much. Therefore, we explain the results of realized GARCH (1, 1) model in what follows.

The distribution of z_t is important in quantile forecasts. We use the standard normal, standardized Student's t- and standardized skewed Student's t-distributions for the standardized error term z_t in (1). The standardized version of the skewed Student's t-distribution introduced by Fernández and Steel (1998) has the pdf:

$$f(z_t|\xi,\upsilon) = \begin{cases} \dfrac{2\xi}{\xi^2+1} sg[\xi(sz_t+m)|\upsilon] & \text{if } z_t < -\dfrac{m}{s}, \\ \dfrac{2\xi}{\xi^2+1} sg[(sz_t+m)/\xi|\upsilon] & \text{if } z_t \geq -\dfrac{m}{s}, \end{cases} \tag{5}$$

where $\upsilon > 2$ and $\xi > 0$. $g[\cdot|\upsilon]$ is the pdf of standardized Student's t-distribution with degree of freedom υ. Parameters m and s^2 are the mean and the variance of the non-standardized skewed Student's t-distribution:

$$m=\frac{\Gamma\left(\frac{\upsilon-1}{2}\right)\sqrt{\upsilon-2}}{\sqrt{\pi}\Gamma\left(\frac{\upsilon}{2}\right)}\left(\xi-\frac{1}{\xi}\right),\quad s^2=\left(\xi^2+\frac{1}{\xi^2}-1\right)-m^2, \tag{6}$$

where ξ and υ determine the skewness and kurtosis, respectively. The skewness of z_t is zero if $\xi = 1$ and positive (negative) if $\xi >$ (<)1. The kurtosis decreases as υ increases.

The likelihood of the realized GARCH model can easily be evaluated as

$$L=\prod_{t=1}^{T}h(R_t|X_1,\dots,X_{t-1})l(X_t|X_1,\dots,X_{t-1},R_t), \tag{7}$$

where $h(R_t|X_1, \dots, X_{t-1})$ is the pdf determined by the distribution of z_t and $l(X_t|X_1, \dots, X_{t-1}, R_t)$ is the normal density with mean $\mu+\varphi\ln\sigma_t^2+\tau_1 z_t+\tau_2(z_t^2-1)$ and variance σ_u^2. Given the initial values σ_0^2 and X_0, we can calculate σ_t^2 by substituting (σ_0^2, $X_1, \dots, X_{t-1}$) sequentially to (2). Hence, it is straightforward to evaluate $h(R_t|X_1, \dots, X_{t-1})$. Given R_t and σ_t, we can calculate $z_t = R_t/\sigma_t$. Thus, it is also straightforward to evaluate $l(X_t|X_1, \dots, X_{t-1}, R_t)$. We set the initial values $\ln\sigma_0^2$ and X_0 equal to the unconditional means $(\omega + \gamma\mu)/(1 - \beta - \gamma\varphi)$ and $\mu+\varphi\ln\sigma_0^2$, respectively. We estimate the degree of freedom υ for the Student's t-distribution or (ξ, υ) for the skewed Student's t-distribution jointly with the parameters (ω, β, γ) in (2) and (μ, φ, τ_1, τ_2, σ_u^2) in (3) by the maximum likelihood method.

3. VaR and expected shortfall

In this article, we concentrate on long position. Then, the 1-day-ahead forecast for the VaR of the daily return R_t with probability α is defined as $\text{VaR}_t(\alpha)$ satisfying

$$\Pr(R_t<\text{VaR}_t(\alpha)|\boldsymbol{I}_{t-1})=\alpha. \tag{8}$$

The sample size of daily returns and realized volatility used in our empirical analysis is 3263. Using 1500 daily returns and realized volatilities, we calculate the 1-day-ahead forecasts ($\text{VaR}_{1501}(\alpha), \dots, \text{VaR}_{3263}(\alpha)$) as follows.

A1. Set $i = 1$.
A2. Estimate the parameters of the realized GARCH model using the sample ($R_i, \dots, R_{1499+i}, X_i, \dots, X_{1499+i}$) by the maximum likelihood method.
A3. Set the parameters (ω, β, γ) in (2) equal to their estimates obtained in A2. Then, calculate σ_{1500+i}^2 by substituting σ_{1499+i}^2 and X_{1499+i} into (2).
A4. Set the parameters υ or (υ, ξ) of the distribution of z_{1500+i} equal to their estimates in A2 if the distribution is Student's t or skewed Student's t. Then, obtain $z_{1500+i}(\alpha)$ satisfying $\Pr(z_{1500+i} < z_{1500+i}(\alpha)) = \alpha$ depending on the distribution.
A5. Set $\text{VaR}_{1500+i}(\alpha) = \sigma_{1500+i}z_{1500+i}(\alpha)$.
A6. Set $i = i+1$ and return to A1 if $i < 1763$ and end if $i = 1763$.

Using ($\text{VaR}_{1501}(\alpha), \dots, \text{VaR}_{3263}(\alpha)$) obtained by executing this algorithm, we calculate the empirical failure rate. Let N be the number of times when the VaR is violated, i.e.,

$R_t < \text{VaR}_t(\alpha)$ for $t = 1501, \ldots, 3263$. Then, the empirical failure rate is defined as $N/1763$. Using the empirical failure rate, we apply the likelihood ratio (LR) test proposed by Kupiec (1995) to test the null hypothesis of $f = \alpha$, where f is the true failure rate. The LR statistic is:

$$\text{LR} = 2\ln\left(\left(\frac{N}{1763}\right)^{N}\left(1-\frac{N}{1763}\right)^{1763-N}\right) - 2\ln\left(\alpha^{N}(1-\alpha)^{1763-N}\right). \tag{9}$$

This LR statistic is asymptotically distributed as a $\chi^2(1)$ if the null hypothesis of $f = \alpha$ is true.

The problem of VaR is that it only measures a quantile of the distribution and hence ignores important information regarding the tails of the distribution beyond this quantile. We also use expected shortfall (ES), which is defined as the conditional expectation of the return given that it is beyond the VaR. The 1-day-ahead forecast for the ES of the daily return R_t with probability α is defined as follows.

$$\text{ES}_t(\alpha) = \text{E}[R_t | R_t < \text{VaR}_t(\alpha), \boldsymbol{I}_{t-1}]. \tag{10}$$

Using $(\text{VaR}_{1501}(\alpha), \ldots, \text{VaR}_{3263}(\alpha))$, we calculate $(\text{ES}_{1501}(\alpha), \ldots, \text{ES}_{3263}(\alpha))$ as follows.

B1. Set $i = 1$.
B2. Simulate 10000 sample for R_{1500+i} using (1) given σ_{1500+i} and the distribution of z_{1500+i}.
B3. Calculate $\text{ES}_{1500+i}(\alpha)$ as the average of the sample violating the VaR, i.e., $R_{1500+i} < \text{VaR}_{1500+i}(\alpha)$.
B4. Set $i = i + 1$ and return to B1 if $i < 1763$ and end if $i = 1763$.

To backtest the predicted ES value with probability α, we use the measure proposed by Embrechts *et al*. (2005). The standard backtesting measure for the ES estimates is

$$D_1(\alpha) = \frac{1}{x(\alpha)}\sum_{t\in\kappa(\alpha)} \delta_t(\alpha), \tag{11}$$

where $\delta_t(\alpha) = R_t - \text{ES}_t(\alpha)$, $x(\alpha)$ is the number of days for which a violation of $\text{VaR}_t(\alpha)$, i.e., $R_t < \text{VaR}_t(\alpha)$ occurs and $\kappa(\alpha)$ is the set of days for which it happens.

Its weakness is that it depends strongly on the VaR estimates without adequately reflecting the correctness of these values. To correct for this, it is combined with the following measure, where the empirical α-quantile of $\delta_t(\alpha)$ is used in place of the VAR estimates.

$$D_2(\alpha) = \frac{1}{y(\alpha)}\sum_{t\in\tau(\alpha)} \delta_t(\alpha), \tag{12}$$

where $y(\alpha)$ is the number of days for which $\delta_t(\alpha)$ is less than its α-quantile and $\tau(\alpha)$ is the set of days for which it happens.

The Embrechts *et al*. (2005) measure is given by

$$D(\alpha) = (|D_1(\alpha)| + |D_2(\alpha)|)/2. \tag{13}$$

A good estimation of ES will lead to a low value of $D(\alpha)$.

4. Empirical application

4.1 Data

We use daily data on returns and realized volatilities of the S&P 500 stock index. The sample period is 3 January 1996–27 February 2009. These data are obtained from the Oxford–Man Institute's Realized Library (Heber *et al*., 2009), where we can download two types of realized volatilities. One is the plain RV, which is the sum of the squared intraday returns. We call it RV in what follows. The other is the realized kernel (RK) calculated using the method proposed by Barndorff-Nielsen *et al*. (2008) to take account of microstructure noise. If the bias of realized volatility caused by microstructure noise can be adjusted by the realized GARCH model, RK will not improve the performance of VaR and ES. To analyze whether this is true, we use the both RV and RK.

Figure 1 plots these data and Table 1 summarizes the descriptive statistics for the full sample. Table 1a shows the descriptive statistics of daily returns (%). The mean is not significantly different from 0. LB(10) is the Ljung-Box statistic adjusted for heteroskedasticity following Diebold (1988) to test the null hypothesis of no autocorrelations up to 10 lags. According to this statistic, the null hypothesis is not rejected at the 10% significance level. Thus, we set $E(R_t|\boldsymbol{I}_{t-1}) = 0$ in (1). The skewness is significantly below 0 and the kurtosis is significantly above 3, indicating the well-known phenomenon that the distribution of the daily return is leptokurtic. The Jarque-Bera (JB) statistic using both skewness and kurtosis also rejects the null hypothesis of normality at the 1% significance level.

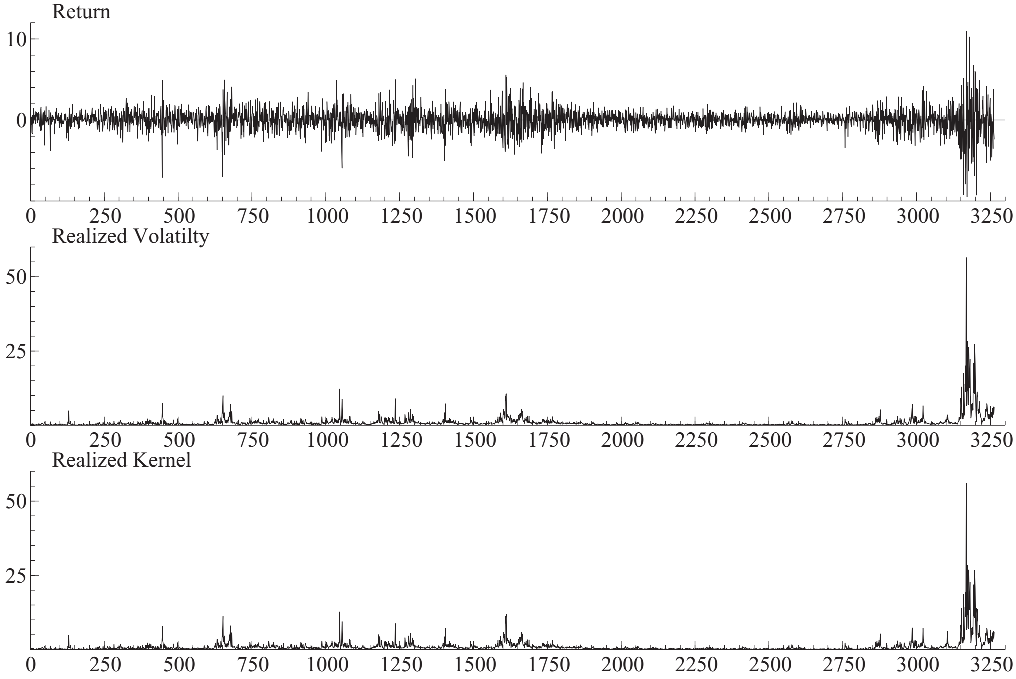

FIGURE 1. Daily rerurns, realized volatility and realized kernel of the S&P 500 stock index

TABLE 1
Descriptive statistics for the full sample

Mean	Standard deviation	Skewness	Kurtosis	JB	LB(10)
		(a) Daily returns (%)			
0.005	1.313	–0.258	11.025	8791.47	16.27
(0.023)		(0.043)	(0.086)		
		(b) Log-realized volatility			
–0.657	0.993	0.556	3.825	260.69	6339.12
(0.017)		(0.043)	(0.086)		
		(c) Log-realized kernel			
–0.621	1.001	0.530	3.772	233.76	6475.19
(0.018)		(0.043)	(0.086)		

Notes: Sample period: 3 January 1996–27 February 2009. Sample size: 3263. The numbers in parentheses are standard errors. JB is the Jarque-Bera statistic to test the null hypothesis of normality. LB(10) is the Ljung-Box statistic adjusted for heteroskedasticity following Diebold (1988) to test the null hypothesis of no autocorrelations up to 10 lags.

Table 1b,c summarizes the descriptive statistics of log-RV and log-RK. The values of skewness, kurtosis and JB statistic indicate that the distributions of log-RV and log-RK are non-normal. Thus, it might be better to assume a non-normal distribution also for u_t in (3), but we leave this for future analysis. LB(10) is so large that the null hypothesis of no autocorrelation is rejected, which is consistent with the phenomenon called volatility clustering.

4.2 Estimation results of the realized GARCH model

As explained in Section 3, we estimate the realized GARCH model using the 1500 daily returns and RV (RK) and then forecast the 1-day-ahead VaR and ES given the parameter estimates. Table 2 summarizes the estimation results of the realized GARCH model with the normal, Student's *t*- and skewed Student's *t*-distributions for z_t using the first 1500 returns and RV (RK). The sample period is from 3 January 1996–4 February 2002.

Table 2a shows the results using RV. Judging from the likelihood values, the skewed Student's *t*-distribution fits the data best. ξ in the skewed Student's *t*-distribution is significantly below 1, indicating a negative skewness of z_t. Figure 2 plots the pdf of the standard normal, standardized Student's *t*- and standardized skewed Student's *t*-distributions where the parameters (υ, ξ) in the skewed Student's *t* and υ in the Student's *t* are set equal to their estimates in Table 2a. The parameter estimates of the realized GARCH model do not depend on the distribution of z_t so much. The persistence in volatility can be measured by the estimates of $\beta + \gamma\varphi$, which is about 0.95 no matter which distribution is used for z_t. This result shows a well-known phenomenon of a high persistence in volatility. The estimate of μ is significantly below 0 and that of φ is significantly above 1 at the 1% significance level, showing that the log-RV is a biased estimator of the true log-volatility. The estimate of τ_1 is significantly below 0, which is consistent with a well-known phenomenon in stock markets of a negative correlation between today's return and tomorrow's volatility (Nelson, 1991). Figure 3 plots the news impact curve, where the horizontal axis is z_{t-1} and the vertical axis is σ_t.

TABLE 2
Estimation results of realized GARCH model for the first 1500 sample

ω	β	γ	μ	φ	τ_1	τ_2	σ_u	υ	ξ
(a) RV									
Normal (Log-likelihood = −3440.53)									
0.258	0.587	0.275	−0.896	1.305	−0.198	0.058	0.518		
(0.024)	(0.031)	(0.022)	(0.051)	(0.076)	(0.014)	(0.007)	(0.009)		
Student's t (Log-likelihood = −3411.36)									
0.261	0.586	0.288	−0.870	1.252	−0.197	0.056	0.518	8.138	
(0.028)	(0.031)	(0.025)	(0.061)	(0.082)	(0.014)	(0.007)	(0.009)	(1.466)	
Skewed Student's t (Log-likelihood = −3397.67)									
0.268	0.590	0.287	−0.920	1.245	−0.197	0.079	0.516	8.625	0.826
(0.028)	(0.030)	(0.024)	(0.061)	(0.076)	(0.014)	(0.009)	(0.009)	(1.670)	(0.029)
(b) RK									
Normal (Log-likelihood = −3453.27)									
0.240	0.589	0.272	−0.843	1.315	−0.197	0.056	0.522		
(0.023)	(0.031)	(0.022)	(0.051)	(0.077)	(0.014)	(0.007)	(0.010)		
Student's t (Log-likelihood = −3424.16)									
0.242	0.588	0.284	−0.817	1.262	−0.196	0.054	0.523	8.142	
(0.027)	(0.031)	(0.025)	(0.062)	(0.084)	(0.015)	(0.007)	(0.010)	(1.481)	
Skewed Student's t (Log-likelihood = −3410.54)									
0.249	0.592	0.284	−0.868	1.258	−0.197	0.077	0.520	8.608	0.826
(0.027)	(0.030)	(0.024)	(0.062)	(0.078)	(0.014)	(0.009)	(0.010)	(1.670)	(0.030)

Notes: Sample period: 3 January 1996–24 February 2002. Sample size: 1500. The numbers in parentheses are standard errors.

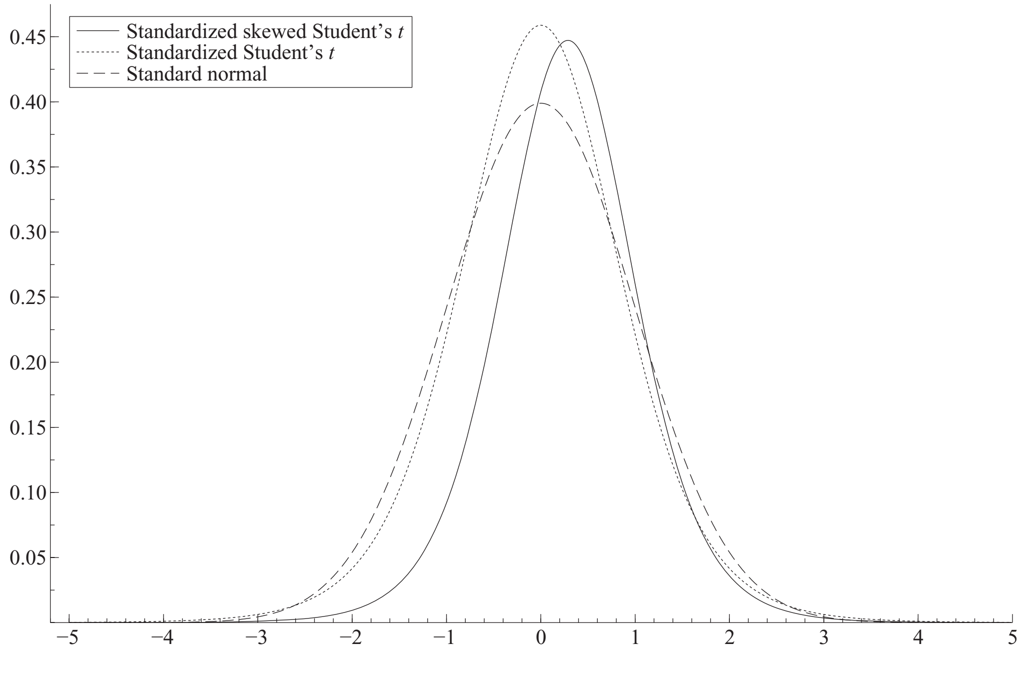

FIGURE 2. Estimated pdf of z_t
*Parameters υ in the Student's t-distribution and (υ, ξ) in the skewed Student's t-distribution are set equal to their estimates in Table 2a.

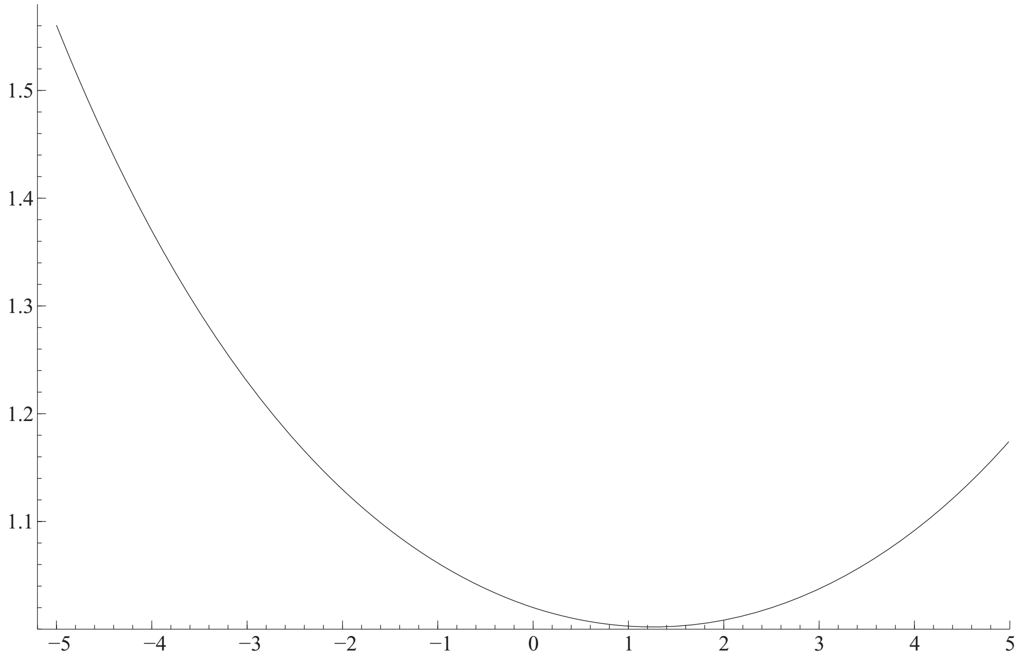

FIGURE 3. News impact curve
*The horizontal axis is z_{t-1} and the vertical axis is σ_t. Parameters are set equal to their estimates in the skewed Studen's t-distribution in Table 2a.

Table 2b summarizes the estimation results of the realized GARCH model using RK. The results in Table 2b are almost the same as those in Table 2a.

For comparison, we also calculate VaR and ES using the EGARCH model proposed by Nelson (1991):

$$\ln\sigma_t^2 = \omega + \phi\left(\ln\sigma_{t-1}^2 - \omega\right) + \theta z_{t-1} + \gamma\left(|z_{t-1}| - E\left(|z_{t-1}|\right)\right). \tag{14}$$

We use the standard normal, the standardized Student's t and the standardized skewed Student's t for the distribution of z_t. Table 3 summarizes the estimation results of the EGARCH model. Judging from the log-likelihood values, the skewed Student's t-distribution fits the data best also in the EGARCH model.

4.3 Comparison using VaR and ES

Table 4a,b shows the empirical failure rates and the P-values for the Kupiec LR test for α = 1%, 5% and 10%, where RG(RV), RG(RK) and EG denote the realized GARCH model with RV, the realized GARCH model with RK and the EGARCH model, and n, t and skt represent the normal, Student's t- and skewed Student's t-distributions for z_t. As can be seen from Table 4b, the null hypothesis of $f = \alpha$ is accepted for all models when α = 5%, 10%. When α = 1%, the null hypothesis is rejected for RG(RV)-n, RG(RK)-n, EG-n and EG-t at the 5% significance level. Thus, we may conclude that it is not good to assume the normal distribution for z_t no matter which model is used. We cannot conclude which model performs best among RG(RV), RG(RK) and EG because

TABLE 3
Estimation results of EGARCH model for the first 1500 sample

ω	ϕ	θ	γ	υ	ξ
		Normal (Log-likelihood = –2289.65)			
0.338	0.948	–0.165	0.101		
(0.067)	(0.011)	(0.019)	(0.020)		
		Student's *t* (Log-likelihood = –2267.98)			
0.280	0.954	–0.158	0.099	8.730	
(0.079)	(0.011)	(0.022)	(0.022)	(1.695)	
		Skewed Student's *t* (Log-likelihood = –2261.16)			
0.297	0.955	–0.155	0.104	9.178	0.871
(0.081)	(0.010)	(0.02)	(0.022)	(1.929)	(0.032)

Notes: Sample period: 3 January 1996–24 February 2002. Sample size: 1500. The numbers in parentheses are standard errors.

TABLE 4
Results for Value-at-Risk

α	10%	5%	1%
	(a) Empirical failure rate		
RG(RV)-*n*	10.323	5.332	1.645
RG(RV)-*t*	10.720	5.672	1.134
RG(RV)-*skt*	10.267	4.594	0.908
RG(RK)-*n*	10.380	5.615	1.645
RG(RK)-*t*	11.004	5.729	1.248
RG(RK)-*skt*	10.323	4.651	0.908
EG-*n*	10.040	5.445	1.872
EG-*t*	10.777	5.559	1.531
EG-*skt*	10.607	5.218	1.248
	(b) *P*-values from the LR test		
RG(RV)-*n*	0.652	0.527	0.013*
RG(RV)-*t*	0.318	0.204	0.579
RG(RV)-*skt*	0.710	0.429	0.692
RG(RK)-*n*	0.597	0.245	0.013*
RG(RK)-*t*	0.166	0.170	0.314
RG(RK)-*skt*	0.652	0.497	0.692
EG-*n*	0.956	0.397	0.001**
EG-*t*	0.282	0.290	0.038*
EG-*skt*	0.400	0.676	0.314

Notes: The numbers in the table are *P*-values from the Kupiec (1995) LR test calculated from the LR statistic (9). * and ** indicate that the null hypothesis of $f = \alpha$ is rejected at the 5% and 1% significance levels, respectively.

the null hypothesis is accepted for all models and α if we use the skewed Student's *t*-distribution.

In Table 5, we show the $D(\alpha)$ values defined by (13). As can be seen from this table, RG(RV)-*skt* gives the lowest values for α = 5%, 1%. For α = 10%, RG(RV)-*t* gives the lowest value but it is not so much different from that of RG(RV)-*skt*. Hence, for the 1-day-ahead ES prediction, the realized GARCH model with the skewed Student's *t*-distribution is superior to that with the other distributions and the EGARCH model.

TABLE 5
Results for expected shortfall

α	10%	5%	1%
RG(RV)-*n*	0.092	0.168	0.437
RG(RV)-*t*	0.070*	0.089	0.237
RG(RV)-*skt*	0.077	0.030*	0.087*
RG(RK)-*n*	0.110	0.174	0.443
RG(RK)-*t*	0.072	0.091	0.223
RG(RK)-*skt*	0.076	0.031	0.120
EG-*n*	0.136	0.240	0.456
EG-*t*	0.082	0.140	0.287
EG-*skt*	0.071	0.064	0.211

Notes: The numbers in the table are the value of $D(\alpha)$ defined by (13). *indicates the lowest value for each α.

RG(RV)-*skt* and RG(RK)-*skt* perform better than EG-*skt* for α = 5%, 1%, indicating that using RV or RK improves the performance. The performance of RG(RV)-*skt* is almost the same as that of RG(RK)-*skt* for α = 10%, 5% and the former is superior to the latter for α = 1%, showing that the realized GARCH model can adjust the bias caused by the microstructure noise and hence we need not take the bias into account in calculating realized volatility.

5. Conclusions and extensions

This article applies the realized GARCH model to quantile forecasts, such as VaR and ES. Using the daily returns, RV and RK of the S&P 500 stock index, we find that the realized GARCH model with the skewed Student's *t*-distribution performs better than that with the normal and Student's *t*-distributions and the EGARCH model using the daily returns only and that the performance does not improve if the RK, which takes account of microstructure noise, is used instead of the plain RV.

Several extensions are possible. First, we used the normal, Student's *t*- and skewed Student's *t*-distributions for z_t. The normal inverse Gaussian (NIG) and generalized hyperbolic (GH) skew Student's *t*-distributions have recently been applied to financial returns (Forsberg and Bollerslev, 2002; Aas and Haff, 2006). It is, however, difficult to estimate the parameters in these distributions by the maximum likelihood method (Aas and Haff, 2006). The joint estimation of the parameters in these distributions and the parameters in the realized GARCH model may be challenging. Using the empirical distribution of the standardized residuals or extreme value theory might improve the performance (Mancini and Trojani, 2011). Second, it is worthwhile using the other realized measures of volatility, such as the realized range (Christensen and Podolskij, 2007; Martens and van Dijk, 2007) and the realized volatility from which significant jumps are removed (Barndorff-Nielsen and Shephard, 2004; Andersen *et al*., 2007). Third, the realized SV model proposed by Takahashi *et al*. (2009) should also be applied to quantile forecasts, for example, using Bayesian estimation via MCMC. This method is time-consuming but enables us to estimate the parameters in the distribution and in the model jointly even if GH skew Student's *t*-distribution is used for z_t (Nakajima and Omori, 2011). It also makes it possible to estimate the parameters and forecast VaR and ES jointly by sampling the parameters and the forecasts of VaR and ES jointly from their posterior distribution.

Final version accepted 28 September 2011.

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